

On the Penrose inequality

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The purpose of this letter is to point out an argument which may ultimately lead to a rigorous proof of the Penrose inequality in the general case. The argument is a variation of Geroch's original proposal for a proof of the positive energy theorem which was later adapted by Jang and Wald to apply to initial data sets containing apparent horizons. The new input is to dispense with the a priori restriction to an initial data set and to use the four-dimensional structure of spacetime in an essential way.

In an attempt to find a counterexample for the cosmic censorship hypothesis Penrose [1] postulated an inequality which relates the area of an apparent horizon to the total mass of an isolated gravitating system. His reasoning, based on what he called the “establishment view” on gravitational collapse, was roughly as follows: consider an asymptotically flat spacetime containing mass/energy at sufficiently high densities that it collapses under its own gravitational attraction so that during the collapse a marginally trapped surface \mathcal{H} and later a trapped surface forms. The singularity theorem implies that the spacetime will develop a singularity in the future. If the cosmic censorship hypothesis is valid, then there will exist an event horizon which encloses the ensuing singularity and the surface \mathcal{H} . Following the “establishment view” on gravitational collapse Penrose argues that the spacetime will ultimately settle down to a Kerr black-hole. Then the area $A_{\mathcal{H}}$ of the apparent horizon \mathcal{H} will be less than the area of the intersection of the event horizon with any spacelike and asymptotically flat hypersurface containing \mathcal{H} . Since the area of the event horizon increases towards the future $A_{\mathcal{H}}$ will also be less than the area of the horizon in the limiting Kerr solution which is given by the well known formula $A_{Kerr} = 8\pi m (m + \sqrt{m^2 - a^2})$ which in turn is less than or equal to the Schwarzschild value $A_S = 16\pi m^2$ for the same mass. This mass, however, cannot exceed the value of the total (ADM) mass M of the system because the Bondi-Sachs mass loss formula implies that during the collapse gravitational radiation had carried away positive energy towards null-infinity. Thus, the final inequality is

$$A_{\mathcal{H}} \leq 16\pi M^2. \quad (1)$$

If one could set up situations in which this inequality was violated then this would provide a strong argument against the validity of the cosmic censorship hypothesis. Although, on the other hand, an independent proof of this inequality would not be a proof of cosmic censorship, there have been numerous attempts to show that the Penrose inequality (1) is, in fact, true without referring to the cosmic censorship hypothesis. One of the most successful attempts has been the argument by Jang and Wald [2] which is based in turn on Geroch's earlier idea [3] to prove the positive energy theorem.

Note, that the inequality (1) relates the area of a finite spacelike 2-surface in spacetime with a quantity defined at infinity. Thus, it is natural to assume that \mathcal{H} is contained in an initial data surface Σ which extends out to (spacelike) infinity. Then the problem of proving (1) becomes a problem within the context of initial data sets for the Einstein equations. The basic idea in [2] was to use the Hawking mass of the 2-surfaces of a foliation of Σ as a quantity which interpolates between the area of the apparent horizon on the one hand and the total mass on the other hand. Geroch [3] had shown that, with some technical assumptions, the rate of change of the Hawking mass along successive 2-surfaces is positive definite provided that the dominant energy condition holds and that the 2-surfaces evolve according to the inverse mean curvature flow. That means that in order to transform one surface into the next each of its points is moved a distance proportional to the inverse of the mean curvature at that point in the direction of the normal at that point. The most limiting assumption in this argument is that the initial data be time symmetric, i.e. its extrinsic curvature in spacetime should vanish. Thus, if the inverse mean curvature flow existed and was smooth then the reasoning in [2] would provide a proof of the Penrose inequality (1) for time symmetric initial data sets. This case, the so called “Riemannian Penrose inequality”, was recently proven rigorously by Huisken and Ilmanen [4,5] by showing the existence of an appropriate weak version of the inverse mean curvature flow which suffices to make the argument work. However, there still remains the fact that Σ has to satisfy additional properties so that the general proof of the Penrose inequality is still lacking.

As noted above, the inequality relates the finite 2-surface \mathcal{H} and infinity. These are the “given data” while the initial hypersurface Σ is added by hand. A priori, no spacelike hypersurface Σ is preferred. This raises the question as to whether it is possible to use the available data to *construct* an appropriate Σ . An indication of how this might be achieved can be found in the work of Israel [6,7] (see also Needham [8]). He considered a trapped 2-surface and showed how to extend it into the future by a 3-dimensional spacelike cylinder whose sections are all trapped so that its interior is causally incarcerated.

Consider a spacelike 2-surface \mathcal{S} embedded in spacetime \mathcal{M}^1 . Associated with \mathcal{S} are two unique null-directions perpendicular to \mathcal{S} . Let l^a and n^a be two null-vectors aligned along these respective null-directions normalized against each other and let ρ and ρ' be the divergences of the respective null-geodesic congruences emanating from \mathcal{S} . Clearly, the null-vectors are only defined up to scale. Under a rescaling $l^a \mapsto cl^a$ the other null-vector and the divergences scale according to $n^a \mapsto (1/c)n^a$, $\rho \mapsto c\rho$ and $\rho' \mapsto (1/c)\rho'$. Thus the combinations $\rho'l^a$ and ρn^a are invariantly defined as are all their linear combinations. Of special interest are the combinations $V^a = \rho'l^a - \rho n^a$ (which was used by Israel and Needham) and $H^a = \rho'l^a + \rho n^a$. In “normal situations” such as for a convex surface in Minkowski space when one of the null congruences diverges while the other converges i.e., when $\rho < 0$ and $\rho' > 0$ then V^a and H^a are timelike and spacelike, respectively. The vector field H^a is, in fact, the *mean curvature vector* of \mathcal{S} in \mathcal{M} . We define the *inverse mean curvature vector* by (with $H^2 = -H^a H_a$)

$$q^a = \frac{1}{H^2} H^a = -\frac{1}{2\rho} l^a - \frac{1}{2\rho'} n^a. \quad (2)$$

We can use q^a on \mathcal{S} to construct another (infinitesimally close) 2-surface \mathcal{S}' according to the rule that we move every point of \mathcal{S} a unit distance along the vector q^a at that point. From \mathcal{S}' we proceed according to the same rule and so on. Thus, we ultimately construct a family of spacelike 2-surfaces \mathcal{S}_λ which are linked along a spacelike vector field q^a with parameter λ to form a spacelike 3-dimensional hypersurface Σ .

Next, consider the Hawking mass [10] associated with the 2-surfaces \mathcal{S}_λ

$$m[\mathcal{S}_\lambda] = \frac{1}{4\pi} \left(\frac{A}{4\pi} \right)^{1/2} \int_{\mathcal{S}_\lambda} [K + \rho\rho'] d^2 S. \quad (3)$$

Here K is the so called complex curvature of the 2-surface \mathcal{S}_λ . As a consequence of the Gauß-Bonnet theorem its integral over the surface evaluates to a real multiple of the Euler characteristic of the surface. We are interested in the rate of change of $m[\mathcal{S}_\lambda]$ along the surfaces. Since we are considering spacelike 2-surfaces we apply the GHP-formalism [11,12]. On each member \mathcal{S}_λ the two null-directions are fixed and we choose null-vectors l^a and n^a along them. The complex spacelike null-vector m^a is tangent to \mathcal{S}_λ and defined up to $m^a \mapsto \gamma m^a$ with $\gamma\bar{\gamma} = 1$. Since m^a is Lie-transported along q^a up to this indeterminacy we have the equation $\mathcal{L}_q m^a = A m^a + B \bar{m}^a$ for some complex-valued functions A and B on \mathcal{S}_λ . This equation can be rewritten in the form

$$q^a \nabla_a m^c = \delta q^c + A m^c + B \bar{m}^c \quad (4)$$

from which we can derive the relationship

$$\delta\rho + \tau\rho + \rho'\kappa = 0 \quad (5)$$

between the spin-coefficients ρ , ρ' , κ and τ together with its complex conjugate and primed versions. The change of the area along the inverse mean curvature vector is given by

$$\frac{d}{d\lambda} A[\mathcal{S}_\lambda] = \frac{d}{d\lambda} \int_{\mathcal{S}_\lambda} d^2 A = \int_{\mathcal{S}_\lambda} \mathcal{L}_q d^2 A = 2A[\mathcal{S}_\lambda], \quad (6)$$

from the definition of the divergences ρ and ρ' .

The change in the Hawking mass is

$$\begin{aligned} \frac{d}{d\lambda} m_H[\mathcal{S}_\lambda] &= m_H[\mathcal{S}_\lambda] \\ &+ \frac{1}{4\pi} \left(\frac{A}{4\pi} \right)^{1/2} \int_{\mathcal{S}_\lambda} [2\rho\rho' + \mathcal{L}_q(\rho\rho')] d^2 A. \end{aligned} \quad (7)$$

¹We will use the notation and conventions of Penrose and Rindler [9] throughout.

From the definition of q^a and the appropriate GHP equations we have²

$$\begin{aligned}\mathcal{L}_q(\rho\rho') &= \rho'q^a\nabla_a\rho + \rho q^a\nabla_a\rho' = -\frac{1}{2}\frac{\rho'}{\rho}\mathfrak{b}\rho - \frac{1}{2}\mathfrak{b}'\rho + \text{“primed”} \\ &= -\frac{1}{2}\frac{\rho'}{\rho}\{\rho^2 + \sigma\bar{\sigma} + \Phi_{00}\} - \\ &\quad -\frac{1}{2}\{\delta'\tau + \rho\rho' + \sigma\sigma' - \tau\bar{\tau} - \Psi_2 - 2\Lambda\} + \text{“primed”}\end{aligned}$$

Here, “primed” indicates the terms obtained from the displayed ones by the priming operation (see [9]). Using the definition of the complex curvature

$$K = \sigma\sigma' - \Psi_2 - \rho\rho' + \Phi_{11} + \Lambda$$

and the Einstein equations in the form

$$\Phi_{ab} = 4\pi G \left(T_{ab} - \frac{1}{4}g_{ab}T^c{}_c \right), \quad \Lambda = \frac{1}{3}\pi G T^c{}_c \quad (8)$$

yields

$$\begin{aligned}\mathcal{L}_q(\rho\rho') &= -3\rho\rho' - K + \frac{1}{2}(-\delta'\tau - \delta\tau' + \tau\bar{\tau} + \tau'\bar{\tau}') + \\ &\quad + \frac{1}{H^2}\{\rho'^2\sigma\bar{\sigma} + \rho^2\sigma'\bar{\sigma}' + 4\pi G T_{ab}V^aV^b\}\end{aligned}$$

Inserting this into (7) and using (5) and its primed version to rewrite the τ -terms in terms of the divergences ρ and ρ' (under the simplifying but not restricting assumption that $\kappa = \kappa' = 0$) gives the final result

$$\begin{aligned}\frac{d}{d\lambda}m_H[\mathcal{S}_\lambda] &= \frac{1}{2}\int_{\mathcal{S}_\lambda}\left\{\frac{\delta\rho}{\rho}\frac{\delta'\rho}{\rho} + \frac{\delta\rho'}{\rho'}\frac{\delta'\rho'}{\rho'}\right\}d^2A \\ &\quad + \int_{\mathcal{S}_\lambda}\frac{1}{H^2}\{\rho'^2\sigma\bar{\sigma} + \rho^2\sigma'\bar{\sigma}'\}d^2A \\ &\quad + 4\pi G \int_{\mathcal{S}_\lambda}\frac{1}{H^2}T_{ab}V^aV^bd^2A\end{aligned} \quad (9)$$

Thus, as long as ρ and ρ' remain non-zero, the rate of change of the Hawking mass along the foliation of 2-surfaces consists of three terms which are manifestly positive provided the dominant energy condition is valid. Note, that the second and third terms are associated with the energy flux penetrating through each particular 2-surface. The third term is the density of the matter as measured by an observer moving along the timelike vector V^a perpendicular to q^a (note that $V^aV_a = H^2$) while the second term can be interpreted as the flux of gravitational wave energy along the two distinguished null-directions [13]. The physical meaning of the first term is unclear. There seems to be a superficial similarity to the expression for the Newtonian gravitational energy density $\partial_a\phi\partial^a\phi$ if the gravitational potential ϕ is formally replaced with $\log\rho$ or $\log\rho'$. This analogy is somewhat supported by explicit examples in Schwarzschild spacetime but this needs to be explored further.

As the 2-surfaces flow out along the inverse mean curvature vector they generate a spacelike hypersurface and the question we need to ask is where this hypersurface will ultimately end up. Will it necessarily become asymptotically Euclidean or could it happen that the hypersurface becomes hyperboloidal i.e. that it bends “up” or “down” just enough to approach null-infinity while remaining spacelike throughout? Since there is no preference in the setup for one or the other time direction this is difficult to imagine and preliminary studies seem to confirm the belief that the hypersurface will become asymptotically Euclidean. However, this is a very difficult question which can probably be answered only by the full existence proof for the flow.

²Since the result of this computation cannot depend on how the null-vectors are extended off the surfaces, we assume that they are geodesic, i.e., $\kappa = \kappa' = 0$ on \mathcal{S}_λ . The calculation without this assumption yields the same result.

Suppose that spacetime contains a marginally trapped surface \mathcal{H} with spherical topology and assume that there is a spacelike hypersurface Σ foliated by 2-surfaces $\{\mathcal{S}_\lambda\}_{\lambda \geq 0}$ obtained by flowing along the inverse mean curvature vector q^a . If Σ is such that $\mathcal{S}_0 = \mathcal{H}$ and is asymptotically Euclidean then

$$\lim_{\lambda \rightarrow \infty} m[\mathcal{S}_\lambda] = M$$

i.e. the limit of the Hawking mass at spacelike infinity is the ADM mass. Furthermore, at the apparent horizon we have $\rho = 0$ and

$$m[\mathcal{H}] = \frac{1}{2} \sqrt{\frac{A}{4\pi}}.$$

Since the Hawking mass never decreases along the inverse mean curvature vector we have $m[\mathcal{H}] \leq M$ or, equivalently, $A_{\mathcal{H}} \leq 16\pi M^2$.

The present argument is very similar to the Jang-Wald argument in that the Hawking mass is used as the principal tool to relate the two quantities in question while an appropriate flow provides the foliation along which the mass increases. The main difference, however, is that here the flow itself defines the spacelike hypersurface. The hypersurface “grows” towards infinity in a manner which is dictated by the energy contents of the spacetime so that the mass never decreases. When viewed within the generated hypersurface Σ , the vector field q^a coincides with (twice) the inverse mean curvature vector of the 2-surfaces \mathcal{S}_λ in Σ . The extrinsic curvature of Σ has the peculiar property that its restriction to any of the leaves \mathcal{S}_λ is tracefree, in other words, the mean curvature of Σ is equal to the negative normal component $K_{ab}r^a r^b$ with r^a the normal to \mathcal{S}_λ . This is the necessary and sufficient condition for writing $\rho\rho' = -p^2/8$ with p the mean curvature of \mathcal{S}_λ in Σ . With this and the relationship $\text{Re}(K) = {}^2R/4$ between the scalar and complex curvatures of the 2-surfaces we obtain

$$\frac{1}{8} [2^2 R - p^2]$$

for the integrand in the mass integral (3) which completely agrees with the expression used by Jang and Wald.

Despite the similarity of the arguments the technical problems seem to be completely different. While the Jang-Wald case was concerned with flowing 2-surfaces in a three-dimensional Riemannian manifold, the present situation has to handle the flow of 2-surfaces in a four-dimensional (Lorentzian) spacetime. Obviously, the present argument does not (yet) provide a rigorous proof of the Penrose inequality but it gives a mathematically beautiful and physically convincing demonstration of the fact that the origin of the inequality lies in the four-dimensional spacetime structure.

The main question which remains to be answered is that of existence and regularity of the flow. This is the same situation as with the Jang-Wald argument which has been made rigorous by Huisken and Ilmanen. Just like in that case there is no reason to assume that the flow will remain smooth. If at any point one of the divergences vanishes then the surface will shoot off with infinite speed towards null-infinity along one of the null-directions. This will result in a discontinuous jump in the flow and the question is whether one can find a formulation for the flow problem which can cope with such situations while still allowing for the above argument. It should be noted also that the starting point of the evolution for the flow is singular because the vector field is singular on the apparent horizon just as in the Riemannian case where the inverse mean curvature vector is singular at the initial minimal surface. However, the hope is that just as in the Riemannian case one can reformulate the flow problem in a way which allows for an appropriate regularization of the flow. Then it might also be possible to start with a trapped surface instead of a marginally trapped one and also to deal with the case when the starting surface has several components.

This work grew out of the attempt to understand the peculiar properties of certain integral formulae [14] which provide a proof of the Penrose inequality in various special cases. This relationship will be discussed in more detail in a future paper.

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